The Littlewood-Richardson rule and the boson-fermion correspondence

1995 J. Phys. A: Math. Gen. 28 L331

(http://iopscience.iop.org/0305-4470/28/12/002)

J. Phys. A: Math. Gen. 28 (1995) L331–L337. Printed in the UK

LETTER TO THE EDITOR

The Littlewood–Richardson rule and the boson–fermion correspondence

T H Baker

Physics Department, University of Tasmania, GPO Box 252C Hobart, Australia 7001

Received 3 January 1995

Abstract. The boson–fermion correspondence is applied to derive explicit formulae for expressing the product of S-functions in terms of sums of S-functions associated to non-standard partitions.

In this letter we show how to derive fairly explicit formulae for the multiplication of two S-functions in terms of S-functions with non-standard partitions by means of the boson–fermion correspondence. We also point out how this method can be used to calculate *skew* S-functions, and also how to extend it to the Hall–Littlewood case. The boson–fermion correspondence (or its generalizations) has been used, for example, in investigating various identities of S-functions [1], Q-functions [2,3] and Hall–Littlewood functions [4] as well as in deriving a procedure for calculating S-function (outer) plethysms [5].

Let us briefly summarize the boson–fermion correspondence as it pertains to S-functions. The algebra $\mathcal{A}$ of free fermions is generated by ψ_i, ψ_i^*, $i \in \mathbb{Z}$ satisfying the anti-commutation relations

$$\{\psi_i, \psi_j\} = 0 = \{\psi_i^*, \psi_j^*\} \qquad \{\psi_i, \psi_j^*\} = \delta_{ij}. \tag{1}$$

There is a Fock representation $\mathcal{F}$ of this algebra with a vacuum $|0\rangle$ which satisfies

$$\psi_i|0\rangle = 0 \ (i < 0) \qquad \psi_i^*|0\rangle = 0 \ (i \geqslant 0).$$

The states in the fermionic Fock space can be naturally associated to S-functions in the following manner: let α_0 and q be operators satisfying $[q, \alpha_0] = \mathrm{i}$ and let them act on the space $\oplus_{k\in\mathbb{Z}} \mathrm{e}^{ikq}$ according to

$$\alpha_0 \mathrm{e}^{ikq} = k\mathrm{e}^{ikq} \qquad \mathrm{e}^{\mathrm{i}q}\mathrm{e}^{\mathrm{i}kq} = \mathrm{e}^{\mathrm{i}(k+1)q}. \tag{2}$$

If $\Lambda(x)$ denotes the space of symmetric polynomials in the indeterminates $(x_1, x_2, \ldots)$, then define vertex operators acting on $\bar{\Lambda} = \Lambda(x) \otimes (\oplus_{k\in\mathbb{Z}} \mathrm{e}^{ikq})$ by

$$\begin{aligned}
\psi(z) &= \exp\left(\sum_{n=1}^{\infty} \frac{p_n(x)}{n} z^n\right) \exp\left(-\sum_{n=1}^{\infty} \frac{\partial}{\partial p_n(x)} z^{-n}\right) \mathrm{e}^{\mathrm{i}q} z^{\alpha_0} \\
\psi^*(z) &= \exp\left(-\sum_{n=1}^{\infty} \frac{p_n(x)}{n} z^n\right) \exp\left(\sum_{n=1}^{\infty} \frac{\partial}{\partial p_n(x)} z^{-n}\right) z^{-\alpha_0} \mathrm{e}^{-\mathrm{i}q}
\end{aligned} \tag{3}$$

where $p_n(x) = \sum_i x_i^n$ are power sum symmetric functions. (For notational simplicity, we drop the symbol $\otimes$ in these and subsequent formulae.) If the modes of these vertex operators are given by the expansion

$$\psi(z) = \sum_{n\in\mathbb{Z}} \psi_n z^n \qquad \psi^*(z) = \sum_{n\in\mathbb{Z}} \psi_n^* z^{-n}$$

then it is well known [6] that the modes ψ_n, ψ_n^* satisfy the anti-commutation relations of the free fermion algebra (1). Moreover, there is an isomorphism $\varrho: \mathcal{F} \to \bar{\Lambda}$ which associates every state $a|0\rangle$, $a \in \mathcal{A}$ in the fermionic Fock space such that $\varrho(|0\rangle) = 1$, and if $0 \leqslant i_s < \cdots < i_1$, $0 < j_r < \cdots < j_1$, then [6]

$$\varrho(\psi_{-j_1}^* \cdots \psi_{-j_r}^* \psi_{i_s} \cdots \psi_{i_1}|0\rangle) = (-1)^{j_1+\cdots j_r + l(l-1)/2} s_\lambda(x) \mathrm{e}^{ilq} \tag{4}$$

where $l = s - r$ and λ is a partition of the form

$$\lambda = (i_1 + 1 - l, i_2 + 2 - l, \ldots, i_s + s - l, r^{j_r - 1}, (r-1)^{j_{r-1} - j_r - 1}, \ldots, 2^{j_2 - j_3 - 1}, 1^{j_1 - j_2 - 1}). \tag{5}$$

In what follows, we shall often ignore the momentum factor e^{ilq} occurring in (4).

Let us begin with the Pieri formula for the multiplication of an S-function by a complete symmetric function $h_n \equiv s_{(n)}$, which takes the form [7]

$$h_n s_\mu = \sum_\lambda s_\lambda \tag{6}$$

where the sum is over all partitions λ such that $\lambda - \mu$ is a horizontal n-strip. That is, the partitions λ occurring in the above product are those obtained by adding n extra boxes to the diagram μ in any manner *provided* that the resulting diagram is a valid diagram, *and* no two of the added boxes lie in the same column.

The question we can ask ourselves is how can we turn multiplication by h_n into an operation involving free fermions? The answer comes from the generating function for h_n which we can write as

$$R(z) = \sum_{p=0}^{\infty} h_p z^p = \exp\left(\sum_{n\geqslant 1} \frac{p_n}{n} z^n\right) = \psi(z)\eta(z)$$

where

$$\eta(z) = \exp\left(\sum_{n\geqslant 1} \frac{\partial}{\partial p_n} z^{-n}\right) z^{-\alpha_0} \mathrm{e}^{-iq}.$$

Thus, when we multiply an S-function, represented by a product of free fermionic currents, by the function h_n, represented by the current $R(z)$, we can shuffle the (annihilation) operator $\eta(z)$ through the currents $\psi(w)$ using the relation

$$\eta(z)\psi(w) = \left(\frac{w/z}{1 - w/z}\right) \psi(m)\eta(z) \tag{7}$$

which will then hit the vacuum, leaving us with an expression involving just free fermions. As an example, let us look at the product $h_k s_{(n,m)}$. We know that $s_{(n,m)} = \varrho(\psi_{n+1}\psi_m|0\rangle)$, so that (dropping the $\varrho(\cdot)$ for simplicity)

$$\begin{aligned} h_k s_{(n,m)} &= \frac{1}{2\pi \mathrm{i}} \oint \frac{\mathrm{d}z\,\mathrm{d}w_1\,\mathrm{d}w_2}{z w_1 w_2} z^{-k} w_1^{-n-1} w_2^{-m} R(z)\psi(w_1)\psi(w_2)|0\rangle \\ &= \frac{1}{2\pi \mathrm{i}} \oint \frac{\mathrm{d}z\,\mathrm{d}w_1\,\mathrm{d}w_2}{z w_1 w_2} z^{1-k} w_1^{-n-1} w_2^{-m} \frac{w_1/z}{1-w_1/z}\frac{w_2/z}{1-w_2/z} \\ &\quad \times \psi(z)\psi(w_1)\psi(w_2)\mathrm{e}^{-\mathrm{i}q}|0\rangle \\ &= \sum_{i,j\geqslant 1} \psi_{k-1+i+j}\psi_{n+1-i}\psi_{m-j}\mathrm{e}^{-\mathrm{i}q}|0\rangle \end{aligned}$$

where the contours in the above integrals circle the origin. The upper limits of this last sum are constrained by the fact that $\mathrm{e}^{-\mathrm{i}q}|0\rangle = \psi^*_{-1}|0\rangle$, so that $\psi_p \mathrm{e}^{-\mathrm{i}q}|0\rangle = 0$ if $p \leqslant -1$. Thus, we finally obtain

$$h_k s_{(n,m)} = \sum_{i=-1}^{n}\sum_{j=0}^{m} s_{(n+m+k-i-j,i,j)} \tag{8}$$

where the expression on the right-hand side involves non-standard S-functions which can be turned into standard S-functions using the modification rules

(i) $s_{(\lambda_1,\ldots,\lambda_i,\lambda_{i+1},\ldots,\lambda_p)} = -s_{(\lambda_1,\ldots,\lambda_{i+1}-1,\lambda_i+1,\ldots,\lambda_p)}$,
(ii) if $\lambda_{i+1} = \lambda_i$ for any i then $s_\lambda = 0$,
(iii) if the last part $\lambda_p < 0$ then $s_{\lambda=0}$.

For k smaller than m or n, many terms on the right-hand side of (8) cancel amongst themselves, so that in this particular case, it is not a very efficient formula. However, we shall soon derive a formula which is more efficient when k is small. This method can be extended to the general case, given by the combinatorial expression (6), with the result being

$$h_k s_{(n_1,\ldots,n_p)} = \sum_{i_1=1}^{n_1+p}\sum_{i_2=1}^{n_2+p-1}\cdots\sum_{i_p=1}^{n_p+1} s_{(k-p+i_1+\cdots+i_p,n_1+1-i_1,\ldots,n_p+1-i_p)}. \tag{9}$$

In the case of the product of a complete symmetric function and a one-hook S-function, we can use the exchange relation

$$\eta(z)\psi^*(w) = \left(\frac{z}{w}-1\right)\psi^*(w)\eta(z)$$

to show that

$$h_n s_{(a|b-1)} = s_{(a+n|b-1)} + s_{(n+a-1|b)} + \sum_{i=1}^{a}(s_{(n+i-1,a-i|b-1,0)} + s_{(n+i-2,a-i|b,0)}). \tag{10}$$

For $n \geqslant a$ the partitions in the above expression are standard, and the result could also have been derived easily by adding n boxes to the hook diagram $(a|b)$ in the prescribed manner.

When $n < a$, the terms on the right-hand side start cancelling each other out. Again, we will be able to derive a more efficient expression for this case.

We would now like to start multiplying S-functions on the left by the two-part S-function $s_{(n,m)}$. The generating function for these S-functions takes the form

$$R(z_1, z_2) = \sum_{n,m\geqslant 0} s_{(n,m)} z_1^n z_2^m = \left(1 - \frac{z_2}{z_1}\right) \exp\left(\sum_{k\geqslant 1} \frac{p_k}{k}(z_1^k + z_2^k)\right).$$

Again, we can decompose this into free fermionic currents and an annihilation operator. Indeed,

$$R(z_1, z_2) = \psi(z_1)\psi(z_2)\eta(z_1, z_2)$$

where

$$\eta(z_1, z_2) = \exp\left(\sum_{n\geqslant 1} \frac{\partial}{\partial p_n}(z_1^{-n} + z_2^{-n})\right) z_1^{-\alpha_0 - 1} z_2^{-\alpha_0} e^{-2iq}.$$

Using the exchange relation

$$\eta(z_1, z_2)\psi(w) = \left(\frac{w/z_1}{1 - w/z_1}\right)\left(\frac{w/z_2}{1 - w/z_2}\right)\psi(w)\eta(z_1, z_2)$$

we can follow the previous example and show that

$$s_{(n,m)} h_k = \sum_{i=1}^{k+1} \sum_{j=1}^{k+2-i} s_{(n+i-1, m+j-1, k+2-i-j)} \tag{11}$$

which is a more efficient expression than (8) for small k. In a similar manner, one can derive the result

$$s_{(n,m)} s_{(p,q)} = \sum_{i_1=1}^{p+2} \sum_{i_2=1}^{p+3-i_1} \sum_{i_3=1}^{q+1} \sum_{i_4=1}^{q+2-s_3} s_{(i_1+i_3+n-2, i_2+i_4+m-2, p+2-i_1-i_2, q+2-i_3-i_4)}. \tag{12}$$

Generally, we can consider multiplication on the left by the general S-function $s_{(n_1,\ldots,n_p)}$ through the use of the generating function

$$R(z_1, \ldots, z_p) = \sum_{n_1,\ldots,n_p\geqslant 0} s_{(n_1,\ldots,n_p)} z_1^{n_1} \cdots z_p^{n_p} = \prod_{i<j}\left(1 - \frac{z_j}{z_i}\right) \exp\left(\sum_{k\geqslant 1} \frac{p_k}{k}(z_1^k + \cdots + z_p^k)\right).$$

This allows us to write

$$R(z_1, \ldots, z_p) = \psi(z_1)\cdots\psi(z_p)\eta(z_1, \ldots, z_p)$$

where

$$\eta(z_1, \ldots, z_p) = \exp\left(\sum_{n\geqslant 1} \frac{\partial}{\partial p_n}(z_1^{-n} + \cdots + z_p^{-n})\right) z_1^{-\alpha_0 - p + 1} \cdots z_p^{-\alpha_0} e^{-piq}.$$

Thus, by using the relation

$$\eta(z_1, \ldots, z_p)\psi(w) = \prod_{j=1}^{p} \left(\frac{w/z_j}{1 - w/z_j} \right) \psi(w)\eta(z_1, \ldots, z_p)$$

we obtain the result

$$s_{(n_1,\ldots,n_p)}h_k = \sum_{i_1=1}^{k+1} \sum_{i_2=1}^{k+2-i_1} \cdots \sum_{i_p=1}^{k+p-i_1-\cdots-i_{p-1}} s_{(i_1+n_1-1, i_2+n_2-1, \ldots, i_p+n_p-1, k+p-i_1-\cdots-i_p)}. \tag{13}$$

By proceeding in the above manner, we can derive the general result

$$s_{(m_1,\ldots,m_q)}s_{(n_1,\ldots,n_p)} = \sum_{\{i\}} s_{\lambda_i} \tag{14}$$

where λ_i is a (non-standard) partition of length $p + q$ of the form

$$\lambda_i = (m_1 - p + i_{11} + i_{21} + \cdots + i_{p1}, \cdots, m_q - p + i_{1q} + \cdots + i_{pq}, n_1 + q - i_{11} - \cdots - i_{1q}, \cdots, n_p + q - i_{p1} - \cdots - i_{pq})$$

and the pq indices i_{ab}, where $1 \leqslant a \leqslant p$, $1 \leqslant b \leqslant q$, in the above sum are subject to the constraints

$$1 \leqslant i_{ab} \leqslant n_a + p - a + b - \sum_{k=1}^{b-1} i_{ak}. \tag{15}$$

When $b = 1$, the finite sum in (15) is taken to be zero. Although this is a very explicit expression for the product of two S-functions in terms of S-functions of non-standard partitions, it is very inefficient when it comes to actual computations. More efficient expressions can be obtained when one switches to Frobenius notation and considers products of hook S-functions.

As an example, let us consider multiplication on the left by one-hook S-functions. They have the generating function

$$H(z; w) = \sum_{n,m\geqslant 0} s_{(n|m-1)} z^n (-w)^m = \frac{1}{1 - z/w} \exp\left(\sum_{k\geqslant 1} \frac{p_k}{k} (z^k - w^k) \right). \tag{16}$$

That is

$$H(z; w) = \psi^*(w)\psi(z)\tau(z; w)$$

where

$$\tau(z; w) = \left(\frac{w}{z} \right)^{\alpha_0} \exp\left(\sum_{k\geqslant 1} \frac{\partial}{\partial p_k} (z^{-k} - w^{-k}) \right).$$

Upon using

$$\tau(z; w)\psi(y) = \left(\frac{w - y}{z - y} \right) \psi(y)\tau(z; w) \tag{17}$$

we see that

$$s_{(n|m-1)}h_p = \sum_{i=0}^{p} s_{(n+1+i,p+1-i,1^{m-2})} + \sum_{i=0}^{p-1} s_{(n+1+i,p-i,1^{m-1})} \tag{18}$$

which is a more efficient version of (10) in the case when p is small.

Note that, until now, we have not found it necessary to use the anti-commutation relations for free fermions. As a final example of the method described above, and one in which the anti-commutation relations are needed, let us consider the product of two one-hook S-functions. By using the generating function (16), along with the exchange relation (17) and

$$\tau(z;w)\psi^*(y) = \left(\frac{z-y}{w-y}\right)\psi^*(y)\tau(z;w) \tag{19}$$

we see that

$$\begin{aligned}(-1)^{m+q}s_{(n|m-1)}s_{(p|q-1)} = \sum_{i,j\geqslant 0}(&\psi^*_{-m-i}\psi_{n+j}\psi^*_{-q+i}\psi_{p-j} - \psi^*_{-m-i}\psi_{n+j+1}\psi^*_{-q+i+1}\psi_{p-j}\\ &- \psi^*_{-m-i-1}\psi_{n+j}\psi^*_{-q+i}\psi_{p-j-1} + \psi^*_{-m-i-1}\psi_{n+j+1}\psi^*_{-q+i+1}\psi_{p-j-1})|0\rangle.\end{aligned}$$

Consider for a moment the first term in the above expression; for it to be non-zero we require $0 \leqslant j \leqslant q$. We also require that *either* $0 \leqslant i \leqslant q-1$ *or* $i = p+q-j$. In this latter case, we must use the relation $\{\psi^*_i, \psi_i\} = 1$ to remove the annihilation operator ψ^*_{p-j}, yielding the term $\sum_{j=0}^{p} \psi^*_{-m-p-q+j}\psi_{n+j}|0\rangle$. After treating the other terms in a similar fashion, and gathering like terms, we obtain

$$\begin{aligned}s_{(n|m-1)}s_{(p|q-1)} = {}& s_{(n+p,m+q-1)} + s_{(n+p+1,m+q-2)} + \sum_{i=1}^{q-1} s_{(n,p|m+i-1,q-i-1)}\\ &+ \sum_{i=1}^{p-1} s_{(n+i+1,p-i-1|m-1,q-1)} + \sum_{j=0}^{p}\sum_{i=0}^{q-2} s_{(n+j+1,p-j|m+i-1,q-i-2)}\\ &+ \sum_{j=0}^{p-1}\sum_{i=0}^{q-1} s_{(n+j,p-j-1|m+i,q-i-1)} + 2\sum_{j=0}^{p-1}\sum_{i=0}^{q-2} s_{(n+j+1,p-j-1|m+i,q-i-2)}.\end{aligned} \tag{20}$$

Let us remark that instead of considering S-function multiplication, we can also consider S-function division (i.e. skewing) and derive similar formulae by considering the generating functions for $D(s_\mu)$. These generating functions will be purely functions of $\partial/\partial p_n$ and, hence, can be applied directly to the generating function for s_λ and crunched together using the standard rules yielding a formula for the skew function $s_{\lambda/\mu}$ in terms of non-standard S-functions, which can then be converted into standard functions using the modification rules.

We can also extend the above considerations to the case of the Hall–Littlewood functions $Q_\lambda(x;t)$. Jing's generalized boson–fermion correspondence [4] allows us to treat the Hall–Littlewood case on exactly the same footing as the S-function case; by following the above method we have, for example,

$$\begin{aligned}Q_{(k)}Q_{(n,m)} = {}& Q_{(k,n,m)} + (1-t)\sum_{p=1}^{n+m} Q_{(k+p,n-p,m)} + (1-t)\sum_{r=1}^{m} Q_{(k+r,n,m-r)}\\ &+ (1-t)^2 \sum_{p=1}^{n+m-1}\sum_{r=1}^{m} Q_{(k+p+r,n-p,m-r)}.\end{aligned} \tag{21}$$

The only caveat to explicit formulae such as (21) is that the non-standard Hall–Littlewood functions occurring on the right-hand side must be modified according to the complicated (compared to the S-function case) rules

$$Q_{\{s,r\}} = \begin{cases} t\,Q_{\{r,s\}} + \sum_{i=1}^{m}(t^{i+1} - t^{i-1})Q_{\{r-i,s+i\}} & r - s = 2m + 1 \\ t\,Q_{\{r,s\}} + \sum_{i=1}^{m-1}(t^{i+1} - t^{i-1})Q_{\{r-i,s+i\}} & \\ \qquad + (t^m - t^{m-1})Q_{\{r-m,s+m\}} & r - s = 2m \end{cases} \tag{22}$$

where by $\{r, s\}$ we mean any partition $(\ldots, r, s, \ldots)$ containing r and s as consecutive elements. In (22) we assume $s < r$ and define $m = [\frac{1}{2}(r - s)]$.

The author takes great pleasure in thanking P D Jarvis for useful discussions.

References

[1] Jarvis P D and Yung C M 1994 *J. Phys. A: Math. Gen.* **27** 903
[2] You Y 1992 *J. Alg.* **145** 349
[3] Jing N 1991 *J. Alg.* **138** 340
[4] Jing N 1991 *Adv. Math.* **87** 226
[5] Baker T H 1994 Symmetric function products and plethysms and the boson–fermion correspondence, UTAS-PHYS-94-12; *J. Phys. A: Math. Gen.* to appear
[6] Jimbo M and Miwa T 1983 *Publ. RIMS, Kyoto University* **19** 943
[7] Macdonald I G 1979 *Symmetric Functions and Hall Polynomials* (Oxford: Oxford University Press)